Bianchi VI_0, VII_0 cosmological models with spin and torsion

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Spatially homogeneous cosmological models of Bianchi types VI_0 and VII_0 based on the Einstein-Cartan theory are considered. Exact solutions are obtained for spinning matter content with a barometric equation of state. They are axisymmetric due to the matter spin, and some are nonsingular.

I. INTRODUCTION

In recent years there has been a growing interest in the Einstein-Cartan theory of spacetime,[1-3] in which the intrinsic spin of matter is incorporated as the source of the torsion of the spacetime manifold. Cosmological models constructed on the basis of the Einstein-Cartan theory have given theoretical support to the Trautman conjecture that the spin-spin interaction implicit in the theory might avert the singularities that characterize general relativity.[4] These have mostly been homogeneous models of Bianchi type I.[5-7] As Tafel has shown that nonsingular models of Bianchi types I–VIII are possible in the context of the Einstein-Cartan theory,[8] it is of interest to have available exact solutions of all types of symmetry.

In this paper we consider spatially homogeneous cosmological models characterized by the Bianchi type VI_0 and VII_0 isometry groups, and diagonal metrics. It is shown that the matter spin imposes axial symmetry about the spin axis. In the VI_0 case, exact solutions are obtained for dust and ultrarelativistic matter, the first of which is nonsingular. As a result of axial symmetry, the type VII_0 models reduce to those of type I, for which various exact solutions are known. The structure of the paper is as follows. In Sec. II an outline is given of the Einstein-Cartan theory in terms of differential forms (see Ref. 1 for details). In Sec. III we set up the field equations for the particular metrics considered, using the Ricci tensor expressions obtained in the Appendix. Section IV contains the solutions as well as a discussion of their basic features.

II. EINSTEIN-CARTAN SPACETIME

In the Einstein-Cartan theory, spacetime is represented by a four-dimensional manifold endowed with a linear connection with nonvanishing torsion and a Lorentz metric. In terms of a set of basis vector fields $\{e_\alpha\}$, $\alpha = 0, 1, 2, 3$, the components $\Gamma^\alpha{}_{\beta\gamma}$ of the connection are determined by the covariant derivative formula

$$\nabla_\alpha e_\beta = \Gamma^\gamma{}_{\beta\alpha} e_\gamma , \qquad (2.1)$$

while the components $T^\alpha{}_{\beta\gamma}$ of the torsion tensor are given by

$$T^\alpha{}_{\beta\gamma} = \Gamma^\alpha{}_{\gamma\beta} - \Gamma^\alpha{}_{\beta\gamma} - C_{\beta\gamma}{}^\alpha . \qquad (2.2)$$

The $C_{\beta\gamma}{}^\alpha$'s measure the extent to which the basis tetrad $\{e_\alpha\}$ is anholonomic, and they can be computed on this basis of the relation

$$d\omega^\alpha = -\tfrac{1}{2} C_{\beta\gamma}{}^\alpha \omega^\beta \wedge \omega^\nu \qquad (2.3)$$

for the exterior derivative of the one-forms ω^α, which are dual to the basis vectors.

It follows from the above defining relations that

$$D\omega^\alpha = d\omega^\alpha + \omega^\alpha{}_\beta \wedge \omega^\beta = T^\alpha = \tfrac{1}{2} T^\alpha{}_{\beta\gamma} \omega^\beta \wedge \omega^\gamma \qquad (2.4)$$

and

$$d\omega^\alpha{}_\beta + \omega^\alpha{}_\gamma \wedge \omega^\gamma{}_\beta = \Omega^\alpha{}_\beta = \tfrac{1}{2} R^\alpha{}_{\beta\gamma\delta} \omega^\gamma \wedge \omega^\delta , \qquad (2.5)$$

where

$$\omega^\alpha{}_\beta = \Gamma^\alpha{}_{\beta\gamma} \omega^\gamma . \qquad (2.6)$$

$R^\alpha{}_{\beta\gamma\delta}$ is the Riemann curvature tensor and D denotes covariant exterior derivative. The latter is equal to the exterior derivative when operating on scalar-valued forms, while on tensor-valued zero-forms it gives their covariant derivative.

Compatibility between the differential and metric structures of the Einstein-Cartan manifold is obtained by demanding that the metric field be covariantly constant, i.e., by

$$Dg_{\alpha\beta} = dg_{\alpha\beta} + \omega_{\alpha\beta} + \omega_{\beta\alpha} = 0 , \qquad (2.7)$$

where

$$\omega_{\alpha\beta} = g_{\alpha\mu} \omega^\mu{}_\beta . \qquad (2.8)$$

Building from the completely antisymmetric tensor $n_{\alpha\beta\gamma\delta} = |\det g_{\alpha\beta}|^{1/2} E_{\alpha\beta\gamma\delta}$, with $E_{0123} = 1$, one can construct the complete set of forms

$$n_{\alpha\beta\gamma} = n_{\alpha\beta\gamma\delta}\omega^\delta , \quad n_{\alpha\beta} = \tfrac{1}{2} \omega^\nu \wedge n_{\alpha\beta\gamma} ,$$

$$n_\alpha = \tfrac{1}{3} \omega^\beta \wedge n_{\alpha\beta}$$

and (2.9)

$$n = \tfrac{1}{4} \omega^\alpha \wedge n_\alpha ,$$

in terms of which the Einstein-Cartan field equations take the concise form

$$\tfrac{1}{2}n_{\alpha\beta\gamma} \wedge \Omega^{\beta\gamma} = -t_\alpha \,, \tag{2.10}$$

$$Dn^\alpha{}_\beta = S^\alpha{}_\beta \,. \tag{2.11}$$

Here, t_α is the stress-energy density three-form, and $S_{\alpha\beta} = -S_{\beta\alpha}$ is the spin density two-form. The units chosen are such that $8\pi G = 1 = C$.

In component form, the field equations read

$$R^\alpha{}_\beta - \tfrac{1}{2}\delta^\alpha{}_\beta R = t^\alpha{}_\beta \,, \tag{2.12}$$

$$T^\alpha{}_{\beta\gamma} - \delta^\alpha{}_\beta T^\mu{}_{\mu\gamma} - \delta^\alpha{}_\gamma T^\mu{}_{\beta\mu} = S^\alpha{}_{\beta\gamma} \,, \tag{2.13}$$

where

$$n_\beta t^\beta{}_\alpha = t_\alpha, \quad n_\gamma S^\nu{}_{\alpha\beta} = S_{\alpha\beta} \,. \tag{2.14}$$

It should be noted that, in general, the Ricci tensor $R_{\alpha\beta}$ is not symmetric, since the geometric structure imposed on the spacetime manifold does not demand it.

The field equations, combined with the Bianchi identities,

$$DT^\alpha = \Omega^\alpha{}_\beta \wedge \omega^\beta, \quad D\Omega^\alpha{}_\beta = 0 \,, \tag{2.15}$$

give rise to conservation equations for the stress-energy and spin densities, which are

$$Dt_\alpha = n(T^\mu{}_{\alpha\nu} t^\nu{}_\mu - \tfrac{1}{2} S^\mu{}_{\lambda\nu} R^{\lambda\nu}{}_{\alpha\mu}) \tag{2.16}$$

and

$$Ds_{\alpha\beta} = t_\beta \wedge \omega_\alpha - t_\alpha \wedge \omega_\beta \,, \tag{2.17}$$

respectively.

III. FIELD EQUATIONS

Assuming a classical description of spin, i.e., the decomposition of the spin tensor $S^\alpha{}_{\beta\gamma}$ as

$$S^\alpha{}_{\beta\gamma} = u^\alpha \sigma_{\beta\gamma} \,,$$

with

$$\sigma_{\alpha\beta} u^\beta = 0 \,, \tag{3.1}$$

where u^α is the fluid velocity, with $u^\alpha u_a = -1$, one finds from the spin-torsion equation (2.13) that

$$T^\alpha{}_{\beta\gamma} = S^\alpha{}_{\beta\gamma} \,, \tag{3.2}$$

while the spin-conservation equation (2.17) yields

$$t_{\alpha\beta} - t_{\beta\alpha} = \nabla_\mu(u^\mu \sigma_{\beta\alpha}) \,. \tag{3.3}$$

In the case of a perfect fluid, the stress-energy tensor $t_{\alpha\beta}$ can be written in the form

$$t_{\alpha\beta} = -h_\alpha u_\beta + p g_{\alpha\beta} \,, \tag{3.4}$$

where h_α is the enthalpy density vector and p is the isotropic pressure. Thus, in this case (3.3) implies that

$$h_\alpha = -(\rho + p)u_\alpha - \nabla_\mu(u^\mu \sigma_{\alpha\nu})u^\nu \,, \tag{3.5}$$

so that (3.4) becomes

$$t_{\alpha\beta} = (\rho + p)u_\alpha u_\beta + p g_{\alpha\beta} + \nabla_\mu(u^\mu \sigma_{\alpha\nu})u^\nu u_\beta \,, \tag{3.6}$$

where $\rho = t_{\alpha\beta} u^\alpha u^\beta$ is the energy density in the rest frame of matter.

We now assume that a perfect fluid of the above type is comoving in a spatially homogeneous universe. The manifold of such a spacetime is invariant under a three-parameter group of isometries, which generates spacelike three-dimensional hypersurfaces. Let the set of one-forms $\{\sigma^i\}$, $i = 1, 2, 3$, span these hypersurfaces. Then

$$d\sigma^i = \tfrac{1}{2} D_{jk}{}^i \sigma^j \wedge \sigma^k \,, \tag{3.7}$$

where the $D_{jk}{}^i$'s are the structure constants of the isometry group.

Taking

$$\begin{aligned} d\sigma^1 &= \sigma^2 \wedge \sigma^3 \,, \\ d\sigma^2 &= e\sigma^1 \wedge \sigma^3 \,, \\ d\sigma^3 &= 0 \,, \end{aligned} \tag{3.8}$$

one has a Bianchi type VI_0 or VII_0 group, for $e = 1$ or -1, respectively.[9]

We assume that the metric is diagonal in the invariant basis, so that the line element has the form

$$ds^2 = -dt^2 + a^2(t)\sigma^1 \otimes \sigma^1 + b^2(t)\sigma^2 \otimes \sigma^2 + c^2(t)\sigma^3 \otimes \sigma^3 \,. \tag{3.9}$$

Then an orthonormal basis is obtainable by letting

$$\begin{aligned} \omega^0 &= dt \,, \\ \omega^1 &= a\sigma^1 \,, \\ \omega^2 &= b\sigma^2 \,, \\ \omega^3 &= c\sigma^3 \,, \end{aligned} \tag{3.10}$$

so that

$$ds^2 = n_{\alpha\beta}\omega^\alpha \otimes \omega^\beta \,, \tag{3.11}$$

where $n_{\alpha\beta}$ is the Minkowski metric.

Next, we take the matter spin to be aligned along the σ^3 direction, which means that

$$T^0 = T^0{}_{12}\omega^1 \wedge \omega^2 = \sigma_{12}\omega^1 \wedge \omega^2 = 2s\omega^1 \wedge \omega^2 \,, \quad T^i = 0 \tag{3.12}$$

where $s = s(t)$.

It follows from (3.6) that

$$t_{\alpha\beta} = \mathrm{diag}(\rho, p, p, p) \tag{3.13}$$

and, using the expressions for the Ricci tensor components obtained in the Appendix, the field equations take the form

$$-R_{00}=(\ln\alpha bc)_{,00}+[(\ln\alpha)_{,0}]^2+[(\ln b)_{,0}]^2+[(\ln c)_{,0}]^2-2s^2=-\tfrac{1}{2}(\rho+3p),$$
$$R_{11}=(\ln\alpha)_{,00}+[(\ln\alpha)_{,0}]^2+(\ln\alpha)_{,0}(\ln bc)_{,0}+\frac{\alpha^4-b^4}{2\alpha^2b^2c^2}=\tfrac{1}{2}(\rho-p),$$
$$R_{22}=(\ln b)_{,00}+[(\ln b)_{,0}]^2+(\ln b)_{,0}(\ln\alpha c)_{,0}+\frac{b^4-\alpha^4}{2\alpha^2b^2c^2}=\tfrac{1}{2}(\rho-p), \tag{3.14}$$
$$R_{33}=(\ln c)_{,00}+[(\ln c)_{,0}]^2+(\ln c)_{,0}(\ln b\alpha)_{,0}-\frac{(\alpha^2+eb^2)^2}{2\alpha^2b^2c^2}=\tfrac{1}{2}(\rho-p),$$
$$R_{12}=-\sigma(\ln sb^2c)_{,0}=0=s(\ln s\alpha^2c)_{,0}=R_{21},$$

where $\alpha_{,0}=d\alpha/dt$, etc.

Equations (3.14) are not all independent, but are connected via the conservation equations for mass-energy and spin. From the projection of (2.16) along u^α we obtain

$$u^\alpha\nabla_\alpha\rho+(\rho+p)\nabla_\alpha u^\alpha=0, \tag{3.15}$$

or, using Eq. (A3) of the Appendix,

$$\rho_{,0}+(\rho+p)(\ln\alpha bc)_{,0}=0. \tag{3.16}$$

Similarly, (3.3) and (3.13) yield

$$\nabla_\alpha(u^\alpha\sigma_{12})=0, \tag{3.17}$$

or

$$s(\ln s\alpha bc)_{,0}=0 \tag{3.18}$$

IV. SOLUTIONS

The last of the field equations (3.14) together with (3.18) imply that

$$s=\frac{s_0}{\alpha bc}, \tag{4.1}$$

where s_0 is constant, and

$$\alpha=b. \tag{4.2}$$

The last relation shows that our models become necessarily axially symmetric, owing to the presence of the spin.

A. Bianchi VII_0 models ($e=-1$)

In this case, axial symmetry implies local rotational symmetry, and the models are also invariant under the Bianchi I group which acts transitively on the hypersurfaces of homogeneity, which are flat.[10] This is demonstrated by writing the metric in the form

$$ds^2=dt^2+\alpha^2(\sigma^1\otimes\sigma^1+\sigma^2\otimes\sigma^2)+c^2\sigma^3\otimes\sigma^3, \tag{4.3}$$

and then choosing

$$\sigma^1=\cos x^3dx^1+\sin x^3dx^2,\quad \sigma^2=-\sin x^3dx^1+\cos x^3dx^1,\quad \sigma^3=dx^3, \tag{4.4}$$

in accord with (3.8) for $e=-1$. Then (4.3) becomes

$$ds^2=-dt^2+\alpha^2[(dx^1)^2+(dx^2)^2]+c^2(dx^3)^2. \tag{4.5}$$

The field equations become

$$2(\ln\alpha)''=\alpha^4c^2(\rho-p)=2(\ln c)'',\quad [(\ln\alpha)']^2+2(\ln\alpha)'(\ln c)'+s_0^2=\alpha^4c^2\rho, \tag{4.6}$$

where the prime denotes differentiation with respect to t' with $dt=\alpha^2cdt'$, and they have been integrated by Kopczynski[5] for the case of dust, and by Kuchowicz[6,7] for other barometric equations of state, where $p=\gamma\rho$, with γ a constant. Raychaudhuri[11] has given a class of solutions when a source-free magnetic field is also present. We quote the results of Kopczynski and Kuchowicz for some representative values of γ for later reference.

The first of (4.6) implies that

$$(\ln\alpha)'-(\ln c)'=\text{a constant}=\Lambda, \tag{4.7}$$

while from (3.16) we obtain

$$\rho R^{3(1+\gamma)}=\text{a constant}=M,$$

with

$$R^3=\alpha^2c. \tag{4.8}$$

Then the second of (4.6) yields

$$(3R_{,0})^2R^4+3s_0^2-\Lambda^2=3MR^{3(1-\gamma)}, \tag{4.9}$$

which is solved to give the following:

(i)

$$R^3=\frac{3M}{4}t^2+\frac{3s_0^2-\Lambda^2}{3M},$$

when $\gamma=0$. When $\Lambda=0$, this reduces to the Trautman dust model.[4]

(ii)

$$R(R^2-\alpha^2)^{1/2}+\alpha^2\cosh^{-1}(R/\alpha)=2\sqrt{M/3}\,t,$$

when $\gamma=\frac{1}{3}$. Here $\alpha^2=(3s_0^2-\Lambda^2)/3M$.

(iii) When $\gamma=1$, $R^3=K^{1/2}t$, unless $K=3M-3s_0^2+\Lambda^2=0$, in which case R^3 is constant, and the universe contracts along the direction of the spin, while it expands in the transverse directions.

The basic feature of these solutions is that, at least in cases (i) and (ii), the singularity at $t=0$ which plagues the corresponding general relativity models is avoided.

B. Bianchi VI_0 ($e=1$)

Now the field equations (3.14) become

$$2(\ln\alpha)''=(1-\gamma)M(\alpha^2c)^{1-\gamma}=2(\ln c)''-4\alpha^4, \tag{4.10a}$$

$$[(\ln\alpha)']^2+2(\ln\alpha)'(\ln c)'+s_0^2-\alpha^4=M(\alpha^2c)^{1-\gamma}, \tag{4.10b}$$

where a barometric equation of state is assumed and the notation is that employed in the VII_0 models above.

For $\gamma=0$, the dust model, we were able to obtain only a particular solution of (4.10). Assuming $c=L\alpha^n$, with L and n constant, one finds that the first of (4.10) is satisfied for $L=4/M$, $n=2$. As M is arbitrary, one can set $M=4$ so that (4.10b) becomes

$$(\alpha_{,0})^2=\frac{1}{\alpha^2}-\frac{s_0^2}{5\alpha^6}. \tag{4.11}$$

Its solution gives

$$R^3=\alpha^2c=\alpha^4=Mt^2+\frac{4s_0^2}{5M}. \tag{4.12}$$

For $\gamma=1$, (4.10a) imples that $(\ln\alpha)'=$ a constant, which we set equal to unity for convenience. Then (4.10b) reads

$$2(\ln c)'=\alpha^4+M-s_0^2-1 \tag{4.13}$$

or

$$\frac{d}{d\tau}(\ln c)=\tau+\frac{M-s_0^2-1}{4\tau},\quad 2\tau=\alpha^2. \tag{4.14}$$

Solving the last equation, one obtains

$$c=\tau^{(M-s_0^2-1)/2}\exp(\tau^2/2), \tag{4.15}$$

so that the line element becomes

$$ds^2=\tau^{(M-s_0^2-1)/2}\exp(\tau^2)(-d\tau^2+\sigma^3\otimes\sigma^3)+2\tau(\sigma^1\otimes\sigma^1+\sigma^2\otimes\sigma^2). \tag{4.16}$$

It is a general feature of the solutions of the Einstein-Cartan equations that they reduce to solutions of the general relativity equations when the torsion vanishes. Thus Eqs. (4.12) and (4.16) with $s_0=0$ give the metric coefficients of general-relativistic Bianchi VI_0 models. The latter have been obtained by Ellis and Mac Callum.[10]

The models presented above are symmetric about the spin axis. This feature was determined by the spin itself, in contrast with the corresponding general-relativistic models, where axial symmetry is usually imposed in order to facilitate the solution of the field equations.

The most important characteristic of these solutions, however, is the nonsingular nature of the dust models. The VI_0 model has a minimum radius $R_{\min}=(\frac{4}{5}s_0^2/M)^{1/3}$ which is slightly smaller than the Trautman model's $R_{\min}=(s_0^2/M)^{1/3}\sim 1$ cm if the universe contains $\sim 10^{80}$ baryons. This value of $R_{\min}$, although a very small size for the universe, is very large compared with the Planck length $L^*=1.6\times 10^{-33}$ cm at which quantum fluctuations are supposed to cause the breakdown of the validity of general relativity (GR). For times $t \gg s_0 \sim 10^{-23}$ sec, on the other hand, the dust models coincide with the corresponding GR models, which shows, according to Trautman,[4] that torsion will have hardly any effect on the hadronic and later phases of development of the universe.

It should be noted, however, that spin was not able to avert the singularities of the ultrarelativistic models. If these models represent a better approximation to the dense stages of the universe than the corresponding dust ones, then the above solutions provide an indication that the introduction of torsion into cosmology cannot solve the problem of the initial singularity.

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APPENDIX

Using Eqs. (3.8) and (3.10) we find

$$\begin{aligned} d\omega^1&=(\ln\alpha)_{,0}\omega^0\wedge\omega^1+\frac{\alpha}{bc}\,\omega^2\wedge\omega^3,\\ d\omega^2&=(\ln b)_{,0}\omega^0\wedge\omega^2+\frac{eb}{\alpha c}\,\omega^1\wedge\omega^3,\\ d\omega^3&=(\ln c)_0\omega^0\wedge\omega^3. \end{aligned} \tag{A1}$$

According to (3.11), the ω^α are orthonormal. Then (2.7) implies that

$$\omega_{\alpha\beta}=-\omega_{\beta\alpha}=-\omega_{\beta\alpha}=\Gamma_{\alpha\beta\gamma}\,\omega^\gamma. \tag{A2}$$

Using (A2) and the fact that the only nonvanishing component of the torsion tensor is, according to (3.12), $T^0{}_{12}=2s$, we find from the structure equation (2.4) that the only nonvanishing Ricci rotation coefficients are

$$\begin{aligned} &\Gamma_{101}=(\ln\alpha)_{,0},\quad \Gamma_{202}=(\ln b)_{,0},\quad \Gamma_{303}=(\ln c)_{,0},\\ &\Gamma_{321}=\Gamma_{312}=\frac{\alpha}{2bc}+\frac{eb}{2\alpha c}=B,\\ &\Gamma_{213}=\left(\frac{\alpha}{2bc}-e\,\frac{b}{2\alpha c}\right)=A,\\ &\Gamma_{012}=s=\Gamma_{201}=\Gamma_{210}, \end{aligned} \tag{A3}$$

and those obtainable from the above using (A2). Substitution of (A3) into (A2) yields

$$\begin{aligned} &\omega^0{}_1=(\ln\alpha)_{,0}\omega^1-s\omega^2=\omega^1{}_2,\quad \omega^1{}_2=-s\omega^0+A\omega^3=-\omega^2{}_1,\\ &\omega^0{}_2=s\omega^1+(\ln b)_{,0}\omega^2=\omega^2{}_0,\quad \omega^1{}_3=-B\omega^2=-\omega^3{}_1,\\ &\omega^0{}_3=(\ln c)_{,0}=\omega^3{}_0,\quad \omega^2{}_3=-B\omega^1=-\omega^3{}_2. \end{aligned} \tag{A4}$$

The second of the structure equations, i.e., (2.5) along with (A1) and (A4), gives the following expressions for the curvature two-form:

$$\Omega^0{}_1 = \{(\ln\alpha)_{,00} + [(\ln\alpha)_{,0}]^2 - s^2\}\,\omega^0\wedge\omega^1 - s(\ln sb^2)_{,0}\,\omega^0\wedge\omega^2 - s\left(A + \frac{eb}{\alpha c}\right)\omega^1\wedge\omega^3$$

$$+\left[\frac{\alpha}{bc}(\ln\alpha)_{,0} - A(\ln b)_{,0} - B(\ln c)_{,0}\right]\omega^2\wedge\omega^3\,,$$

$$\Omega^0{}_2 = \{(\ln b)_{,00} + [(\ln b)_{,0}]^2 - s^2\}\,\omega^0\wedge\omega^2 + s(\ln s\alpha^2)_{,0}\,\omega^0\wedge\omega^1 - s\left(A - \frac{\alpha}{bc}\right)\omega^2\wedge\omega^3$$

$$+\left[A(\ln\alpha)_{,0} + \frac{eb}{\alpha c}(\ln b)_{,0} - B(\ln c)_{,0}\right]\omega^1\wedge\omega^3\,,$$

$$\Omega^0{}_3 = B(\ln b\alpha^{-1})_{,0}\,\omega^1\wedge\omega^2 + \{(\ln c)_{,00} + [(\ln c)_{,0}]^2\}\,\omega^0\wedge\omega^3\,,$$

$$\Omega^1{}_2 = A(\ln cA)_{,0}\,\omega^0\wedge\omega^3 + [s^2 + B^2 + (\ln\alpha)_{,0}(\ln b)_{,0}]\,\omega^1\wedge\omega^2\,,$$

$$\Omega^1{}_3 = -B(\ln bB)_{,0}\,\omega^0\wedge\omega^2 + sB\omega^0\wedge\omega^1 - s(\ln c)_{,0}\,\omega^1\wedge\omega^3 + \left[AB - \frac{ebB}{\alpha c} + (\ln\alpha)_{,0}(\ln c)_{,0}\right]\omega^1\wedge\omega^3\,,$$

$$\Omega^2{}_3 = -B(\ln\alpha B)_{,0}\,\omega^0\wedge\omega^1 - sB\omega^0\wedge\omega^2 + s(\ln c)_{,0}\,\omega^1\wedge\omega^3 - \left[AB + \frac{\alpha B}{bc} - (\ln b)_{,0}(\ln c)_{,0}\right]\omega^2\wedge\omega^3\,.$$

Comparison of these expressions with $\Omega^\alpha{}_\beta = \frac{1}{2}R^\alpha{}_{\beta\gamma\delta}\,\omega^\gamma\wedge\omega^\delta$ yields the curvature tensor components, which, when substituted in the defining relation

$$R^\alpha{}_\beta = R^{\mu\alpha}{}_{\mu\beta}\,, \tag{A5}$$

give the Ricci tensor coefficients employed in (3.14).

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